

On the Orthogonal Basis of the Symmetry Classes of Tensors Associated with Certain Characters

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Abstract

A necessary condition is given for the existence of an O-basis for the symmetry classes of tensors associated with a finite group and the irreducible constituents of the permutation character of the finite group. This extends a result of R. R. Holmes [see the Main theorem of Linear and Multilinear Algebra 39 (1995), 241-243].

Keywords: Symmetry class of tensors, Orthogonal basis, Permutation character, Irreducible constituent.

2000 Mathematics Subject Classification: Primary 20C30; Secondary 15A69.

1 Introduction

Let V be an m -dimensional vector space over the complex field \mathbb{C} . Let nV be the n th tensor power of V and write $v_1 \otimes \cdots \otimes v_n$ for the decomposable tensor product of the indicated vectors. To each permutation σ in \mathfrak{S}_n there corresponds a unique linear operator $P(\sigma): {}^nV \rightarrow {}^nV$ determined by $P(\sigma)(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$. Let G be a subgroup of \mathfrak{S}_n and let $\text{Irr}(G)$ be the set of all the irreducible complex characters of G . It follows from the orthogonality relations for characters that

$$\left\{ T(G, \chi) : {}^nV \rightarrow {}^nV \mid T(G, \chi) = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) P(\sigma), \chi \in \text{Irr}(G) \right\}$$

is a set of annihilating idempotents which sum to the identity. The image of nV under the $T(G, \chi)$ is called the *symmetry class of tensors* associated with G and χ and is denoted by $V_\chi^n(G)$. The image of $v_1 \otimes \cdots \otimes v_n$ under $T(G, \chi)$ is denoted by $v_1 * \cdots * v_n$ and is called a *decomposable symmetrized tensor*.

Let Γ_m^n be the set of all sequences $\alpha = (\alpha_1, \dots, \alpha_n)$ with $1 \leq \alpha_i \leq m$. Then the group G acts on Γ_m^n by $\sigma \cdot \alpha = (\alpha_{\sigma^{-1}(1)}, \dots, \alpha_{\sigma^{-1}(n)})$, where $\sigma \in G$ and $\alpha \in \Gamma_m^n$. Let $O(\alpha) = \{\sigma \cdot \alpha \mid \sigma \in G\}$ be the *orbit* of α , and G_α be its *stabilizer subgroup*, i.e., $G_\alpha = \{\sigma \in$

*This research was in part supported by a grant from IPM.

$G \mid \sigma \cdot \alpha = \alpha$ }, and consider a system Δ of distinct representatives of the orbits of Γ_m^n .

Let $\{e_1, \dots, e_m\}$ be a basis of V . Denote by e_α^* the tensor $e_{\alpha_1} * \dots * e_{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in \Gamma_m^n$. For $\alpha \in \Delta$, $V_\alpha^* = \langle e_{\sigma \cdot \alpha}^* \mid \sigma \in G \rangle$ is called the *orbital subspace* of $V_\chi^n(G)$, and we can easily prove that

$$V_\chi^n(G) = \bigoplus_{\alpha \in \Delta} V_\alpha^*. \quad (1)$$

Note that it is possible for some $\alpha \in \Delta$ to have $V_\alpha^* = 0$. But Freese (see [2]) proved that for $\alpha \in \Delta$

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma), \quad (2)$$

therefore, if we consider

$$\bar{\Delta} = \left\{ \alpha \in \Delta \mid \sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0 \right\},$$

then the sum in (1) will be reduced to the form

$$V_\chi^n(G) = \bigoplus_{\alpha \in \bar{\Delta}} V_\alpha^*. \quad (3)$$

Of course we define the right-hand side of (3) to be 0, if $\bar{\Delta} = \emptyset$.

A particular case appears when we assume that V is an m -unitary space. In this case, the inner product on V induces an inner product on $\otimes^n V$, whose restriction to $V_\chi^n(G)$ satisfies

$$\langle u_1 * \dots * u_n \mid v_1 * \dots * v_n \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^n A_{i\sigma(i)},$$

where $A = [A_{ij}]_{n \times n} = [\langle u_i \mid v_j \rangle]_{n \times n}$.

If $\{e_1, \dots, e_m\}$ is an orthonormal basis of V , then the sum appeared in (3) is an orthogonal direct sum. In addition, we can obtain

$$\langle e_{g \cdot \alpha}^* \mid e_{g' \cdot \alpha}^* \rangle = \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi(g' \sigma g^{-1}). \quad (4)$$

Let W be a subspace of $V_\chi^n(G)$. An orthogonal basis of W of the form $\{e_\alpha^* \mid \alpha \in S\}$, where S is a subset of Γ_m^n , is called an *O-basis* of W . Because the sum appeared in (3) is an orthogonal direct sum, $V_\chi^n(G)$ has an O-basis if and only if for all $\alpha \in \bar{\Delta}$, the orbital subspace V_α^* has an O-basis. Note that, if χ is of degree one, since $\dim V_\alpha^* = 1$ for all

$\alpha \in \overline{\Delta}$, then V_α^* has an O-basis for all $\alpha \in \overline{\Delta}$, which implies that $V_\chi^n(G)$ has such a basis.

Several papers are devoted to the investigation of the existence of an O-basis for $V_\chi^n(G)$, see for example [6]. In [4] a necessary and sufficient condition for the existing of an O-basis for $V_\chi^n(G)$ is given, where G is a cyclic or a dihedral group and in [1] when G is a dicyclic group.

The main purpose of this article is to generalize a result of Holmes (see [3, Main theorem]). In fact, he proved that if G is a 2-transitive subgroup of \mathbb{S}_n , $n \geq 3$, $\chi = \theta - 1_G$, where θ is permutation character of G , and $m \geq 2$, then $V_\chi^n(G)$ does not have an O-basis.

In the following section, we will omit condition of “2-transitivity of G ” and will find a necessary condition for the existence of an O-basis for the symmetry class of tensors associated with G and any constituent of its permutation character. This generalization will be independent of the permutation structure of the group G .

2 Results

Let V be an m -unitary space, G a finite group, and Ω a set of n elements. Suppose G acts faithfully on Ω , so we can assume that G is a subgroup of \mathbb{S}_n . In fact, we consider $\{f_\sigma \mid \sigma \in G\}$ as the group G , where $f_\sigma : \Omega \rightarrow \Omega$ defined by $f_\sigma(\omega) = \sigma \cdot \omega$ for all $\omega \in \Omega$, is a permutation on n letters. Therefore the inner product space $V_\chi^n(G)$ is meaningful for all $\chi \in \text{Irr}(G)$. Denote the permutation character of G by θ . For $\sigma \in G$, the value $\theta(\sigma)$ is the number of letters fixed by σ , i.e., $\theta(\sigma) = |\{\omega \in \Omega \mid \sigma \cdot \omega = \omega\}|$.

Main Theorem *Let G be a finite group and let Ω be a set of n elements, $n \geq 2$. Assume that G acts transitively and faithfully on Ω and let V be an m -unitary space, $m \geq 2$. Let χ be an irreducible constituent of θ , where θ is the permutation character of G . If $\chi(1)(\chi, \theta)_G > \frac{1}{2}n$, then $V_\chi^n(G)$ does not have an O-basis.*

Proof. Let $\{e_1, \dots, e_m\}$ be an orthonormal basis of V . Suppose $V_\chi^n(G)$ has an O-basis. Then, by (3), for all $\alpha \in \overline{\Delta}$, the orbital subspace V_α^* has an O-basis. Put $\alpha = (1, 2, \dots, 2)$, then $n \geq 2$ and $m \geq 2$ implies that $\alpha \in \Gamma_m^n$. Consider the action of G on Γ_m^n and choose Δ such that $\alpha \in \Delta$.

Without loss of generality, we can assume that $\Omega = \{1, 2, \dots, n\}$ and therefore $G_\alpha = G_1$, where G_α is the stabilizer subgroup of α when G acts on Γ_m^n and G_1 is the stabilizer subgroup of 1 when G acts on Ω . So by Lemma 5.14 of [5] we obtain $1_{G_\alpha} \uparrow^G = 1_{G_1} \uparrow^G = \theta$.

Frobenius reciprocity implies that

$$\sum_{\sigma \in G_\alpha} \chi(\sigma) = |G_\alpha|(\chi \downarrow_{G_\alpha}, 1_{G_\alpha})_{G_\alpha} = |G_\alpha|(\chi, 1_{G_\alpha} \uparrow^G)_G = |G_\alpha|(\chi, \theta)_G, \quad (5)$$

so by hypotheses, $(\chi, \theta)_G \neq 0$, and we obtain $\sum_{\sigma \in G_\alpha} \chi(\sigma) \neq 0$ and $\alpha \in \bar{\Delta}$. By the above discussion, V_α^* , $\alpha = (1, 2, \dots, 2)$, has an O-basis. We now have $[G : G_\alpha] = [G : G_1] = |\Omega| = n$, so $G = \bigcup_{i=1}^n g_i G_\alpha$, where $\{g_1, \dots, g_n\}$ is a system of distinct representatives of left cosets of G_α in G .

Since, by (2) and (5),

$$\dim V_\alpha^* = \frac{\chi(1)}{|G_\alpha|} \sum_{\sigma \in G_\alpha} \chi(\sigma) = \chi(1)(\chi, \theta)_G =: s,$$

we can assume that $\{e_{g_1 \cdot \alpha}^*, e_{g_2 \cdot \alpha}^*, \dots, e_{g_s \cdot \alpha}^*\}$ is an O-basis for V_α^* .

Define the $n \times n$ complex matrix $A = [A_{ij}]$ by

$$A_{ij} = \langle e_{g_i \cdot \alpha}^* | e_{g_j \cdot \alpha}^* \rangle.$$

Note that $s < n$, therefore A may be written in the form

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_1, A_2, A_3 , and A_4 are matrices of sizes $s \times s, s \times (n-s), (n-s) \times s$, and $(n-s) \times (n-s)$, respectively. But, by (4), for $i, j, 1 \leq i, j \leq s$, we have

$$\begin{aligned} A_{ij} &= \langle e_{g_i \cdot \alpha}^* | e_{g_j \cdot \alpha}^* \rangle \\ &= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi(\sigma) & \text{if } i = j, \end{cases} \\ &= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{1}{[G:G_\alpha]} \dim V_\alpha^* & \text{if } i = j, \end{cases} \\ &= \begin{cases} 0 & \text{if } i \neq j, \\ \frac{s}{n} & \text{if } i = j, \end{cases} \\ &= \left(\frac{s}{n} I_s\right)_{ij}, \end{aligned}$$

where I_s is the $s \times s$ identity matrix. So the block form of A has been changed to the form

$$A = \begin{bmatrix} \frac{s}{n} I_s & A_2 \\ A_3 & A_4 \end{bmatrix}.$$

We now claim that A is idempotent. In fact,

$$\begin{aligned} (A^2)_{ij} &= \sum_{\ell=1}^n A_{i\ell} A_{\ell j} \\ &= \sum_{\ell=1}^n \langle e_{g_i \cdot \alpha}^* | e_{g_\ell \cdot \alpha}^* \rangle \langle e_{g_\ell \cdot \alpha}^* | e_{g_j \cdot \alpha}^* \rangle \\ &= \sum_{\ell=1}^n \left(\frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi(g_\ell \sigma g_i^{-1}) \right) \left(\frac{\chi(1)}{|G|} \sum_{\tau \in G_\alpha} \chi(g_j \tau g_\ell^{-1}) \right) \\ &= \frac{\chi(1)^2}{|G|^2} \sum_{\ell=1}^n \sum_{\sigma \in G_\alpha} \sum_{\tau \in G_\alpha} \chi(g_\ell \sigma g_i^{-1}) \chi(g_j \tau g_\ell^{-1}) \\ &= \frac{\chi(1)^2}{|G|^2} \sum_{\ell=1}^n \sum_{\lambda \in g_\ell G_\alpha} \sum_{\mu \in G_\alpha g_\ell^{-1}} \chi(\lambda g_i^{-1}) \chi(g_j \mu) \\ &= \frac{\chi(1)^2}{|G|^2} \sum_{\ell=1}^n \sum_{\lambda \in g_\ell G_\alpha} \sum_{\sigma \in G_\alpha} \chi(\lambda g_i^{-1}) \chi(g_j \sigma \lambda^{-1}) \\ &= \frac{\chi(1)^2}{|G|^2} \sum_{\lambda \in G} \sum_{\sigma \in G_\alpha} \chi(\lambda g_i^{-1}) \chi(g_j \sigma \lambda^{-1}) \\ &= \frac{\chi(1)^2}{|G|^2} \sum_{\lambda \in G} \sum_{\sigma \in G_\alpha} \chi(\lambda) \chi(g_j \sigma g_i^{-1} \lambda^{-1}) \\ &= \frac{\chi(1)^2}{|G|^2} \sum_{\sigma \in G_\alpha} \sum_{\lambda \in G} \chi(\lambda) \chi(g_j \sigma g_i^{-1} \lambda^{-1}) \\ &= \frac{\chi(1)^2}{|G|^2} \sum_{\sigma \in G_\alpha} \left(\frac{|G|}{\chi(1)} \chi(g_j \sigma g_i^{-1}) \right) \\ &= \frac{\chi(1)}{|G|} \sum_{\sigma \in G_\alpha} \chi(g_j \sigma g_i^{-1}) \end{aligned}$$

$$= \langle e_{g_i \cdot \alpha}^* | e_{g_j \cdot \alpha}^* \rangle$$

$$= A_{ij}.$$

Therefore, the claim holds. Now, using $A^2 = A$, we obtain

$$A_2 A_3 = \left(\frac{s}{n} - \frac{s^2}{n^2} \right) I_s.$$

Since $0 < s < n$, $A_2 A_3$ is an invertible matrix and hence $s \leq n-s$, $s \leq \frac{1}{2}n$, or $\chi(1)(\chi, \theta)_G \leq \frac{1}{2}n$, which is a contradiction. Thus $V_\chi^n(G)$ does not have an O-basis. \square

Corollary 1 (Main theorem [3]) *Let G be a 2-transitive subgroup of \mathcal{S}_n , $n \geq 3$. Let $\chi = \theta - 1_G$, where θ is the permutation character of G . If $\dim V = m \geq 2$, then $V_\chi^n(G)$ does not have an O-basis.*

Proof. G has a natural transitive and faithful action on the set $\Omega = \{1, 2, \dots, n\}$, given by $\sigma \cdot i = \sigma(i)$. 2-Transitivity of this action implies that χ is an irreducible constituent of θ . On the other hand, $n \geq 3$ and so $\chi(1)(\chi, \theta)_G = (n-1) > \frac{1}{2}n$. Hence, by our main theorem, $V_\chi^n(G)$ does not have an O-basis. \square

Example Let $G = \Omega = \mathbb{A}_4$ be the alternating group of degree 4. We know that G acts transitively and faithfully on Ω by right multiplication, so G is a subgroup of \mathcal{S}_{12} . G has an irreducible complex character, say χ , of degree 3. As θ , the permutation character of G , is regular, χ is an irreducible constituent of θ of multiplicity 3. Therefore, $\chi(1)(\chi, \theta)_G = 9 > 6 = \frac{1}{2}|\Omega|$, so by the main theorem $V_\chi^{12}(G)$ does not have an O-basis. Note that in this example the action of G on Ω is not 2-transitive, so our main result really extends the result of [3].

References

- [1] M. R. Darafsheh, M. R. Pournaki, *On the Orthogonal Basis of the Symmetry Classes of Tensors Associated with the Dicyclic Group*, Linear and Multilinear Algebra **47** (2000), no. 2, 137-149.
- [2] R. Freese, *Inequalities for Generalized Matrix Functions Based on Arbitrary Characters*, Linear Algebra Appl. **7** (1973), 337-345.

- [3] R. R. Holmes, *Orthogonal Bases of Symmetrized Tensor Space*, Linear and Multilinear Algebra **39** (1995), 241-243.
- [4] R. R. Holmes, T. Y. Tam, *Symmetry Classes of Tensors Associated with Certain Groups*, Linear and Multilinear Algebra **32** (1992), 21-31.
- [5] I. M. Isaacs, “*Character Theory of Finite Groups*”, Academic Press, New York, 1976.
- [6] B. Y. Wang, M. P. Gong, *The Subspace and Orthonormal Bases of Symmetry Classes of Tensors*, Linear and Multilinear Algebra **30** (1991), 195-204.

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